Chapter 6 – Renewal Processes

1 Definition of a Renewal Process

We have shown that the interarrival times for the Poisson process are independent and identically distributed exponential random variables. A natural generalization is to consider a counting process for which the interarrival times are independent and identically distributed with an arbitrary distribution. Such a counting process is called a *Renewal Process*.

Let $\{X_n, n = 1, 2, \cdots\}$ be a sequence of positive independent random variables with a common cumulative distribution $F_X(\cdot)$. Let $W_n = \sum_{i=1}^{n} X_i$, $n \geq 1$ and define $N(t) = \sup\{n : W_n \leq t\}$. The process $\{N(t), t \geq 0\}$ is a renewal process. We say that a renewal process occurs at *t* if $W_n = t$ for some $n \geq 1$. Since the interarrival times are independent and identically distributed, it follows that after each renewal the process starts over again.

The principal objective of renewal theory is to derive properties of certain random variables associated with $\{N(t)\}$ and $\{W_n\}$ from knowledge of the inter-occurrence distribution $F_X(\cdot)$.

<u>Renewal function</u>

An important relationship for the renewal process is

$$N(t) \geq n \quad \text{if and only if} \quad W_n \leq t \tag{6-1}$$

From (6-1), we obtain

$$\begin{aligned}
P[N(t) = n] &= P[N(t) \leq n] - P[N(t) \leq n-1] \\
&= F_{N(t)}(n) - F_{N(t)}(n-1) \\
&= \{1 - P[N(t) > n]\} - \{1 - P[N(t) > n-1]\} \\
&= P[N(t) > n-1] - P[N(t) > n] \\
&= P[N(t) \geq n] - P[N(t) \geq n+1] \\
&= P[W_n \leq t] - P[W_{n+1} \leq t] \\
&= F_{W_n}(t) - F_{W_{n+1}}(t)
\end{aligned}$$

Let the expected number of arrivals for the time duration (0, *t*], also called the *renewal function*, be

$$E[N(t)] = m(t). \tag{6-2}$$

$$m(t) = \sum_{n=1}^{\infty} F_{W_n}(t) \qquad (6\text{-}3)$$

[Proof]

$$N(t) = \sum_{n=1}^{\infty} A_n$$

where

$$A_n = \begin{cases} 1 & \text{if the } n\text{th arrival occurs in } [0,t] \\ 0 & \text{otherwise} \end{cases}$$

Hence,

$$E[N(t)] = E\left\{\sum_{n=1}^{\infty} A_n\right\} = \sum_{n=1}^{\infty} E[A_n] = \sum_{n=1}^{\infty} P[A_n = 1]$$

$$= \sum_{n=1}^{\infty} P[W_n \le t]$$

$$= \sum_{n=1}^{\infty} F_{W_n}(t).$$

[Theorem]

$$E[W_{N(t)+1}] = E[X_1]E[N(t)+1] \qquad (6\text{-}4)$$

$$E[W_{N(t)+1}] = E[X_1 + \cdots + X_{N(t)+1}]$$

$$= E[X_1] + E[\sum_{j=2}^{N(t)+1} X_j] = \mu + E\left[\sum_{j=2}^{\infty} X_j \mathbf{I}\{N(t)+1 \ge j\}\right]$$

$$N(t) \ge j-1 \text{ if and only if } X_1 + \cdots + X_{j-1} \le t,$$

$$\mathbf{I}\{N(t) \ge j-1\} = \mathbf{I}\{X_1 + \cdots + X_{j-1} \le t\}.$$

$$E\left[X_j \mathbf{I}\{X_1 + \cdots + X_{j-1} \le t\}\right] = E[X_j]E\left[\mathbf{I}\{X_1 + \cdots + X_{j-1} \le t\}\right]$$

$$= E[X_j]P\{X_1 + \cdots + X_{j-1} \le t\}$$

$$= \mu F_{W_{j-1}}(t) \qquad \left(E[X_j] = \mu\right)$$

Therefore,

$$E\left[W_{N(t)+1}\right] = \mu + \sum_{j=2}^{\infty} E\left[X_j \mathbf{I}\left\{X_1 + \cdots + X_{j-1} \le t\right\}\right]$$

$$= \mu + \mu \sum_{j=2}^{\infty} F_{j-1}(t)$$

$$= \mu\left[1 + M(t)\right]$$

$$E\left[W_{N(t)+1}\right] = \mu\left[1 + m(t)\right] \qquad (6\text{-}5)$$

<u>**The Poisson process viewed as a renewal process**</u>

The Poisson process with parameter α is a renewal process whose interarrival times have the exponential distribution $F_X(x) = 1 - e^{-\alpha x}$, $x \ge 0$.

- **The renewal function**

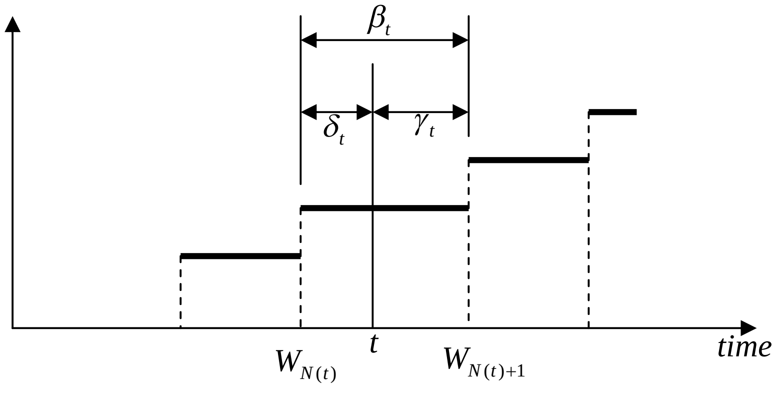
  $m(t) = E[N(t)] = \alpha t$.

There are some random variables that are of interest in renewal theory. Three of these are the *excess life*, the *current life* and the *total life*, defined, respectively, by

$\gamma_t = W_{N(t)+1} - t$ (excess or residual lifetime)

$\delta_t = t - W_{N(t)}$ (current life or age random variable)

$\beta_t = \gamma_t + \delta_t$ (total life)



- **Excess life**

  The excess life time at time *t* exceeds *x* if and only if there are no renewals in the interval $(t, t+x]$. This event has the same probability as that of no renewals in the interval $(0, x]$, since a Poisson process has stationary independent increment, i.e.,

  $P[\gamma_t > x] = P[N(t+x) - N(t) = 0]$

  $= P[N(x) = 0] = e^{-\alpha x}$

Thus, in a Poisson process, the excess life possesses the same exponential distribution

$$P[\gamma_t \le x] = 1 - e^{-\alpha x}, \quad x \ge 0.$$

as every life.

- **Current life**

  The current life δ_t can not exceed *t*, while for $x < t$ the current life exceeds *x* if and only if there are no renewals in $(t - x, t]$, which again has probability $e^{-\alpha x}$. Thus, the current life follows the truncated exponential distribution

$$P[\delta_t \le x] = \begin{cases} 1 - e^{-\alpha x} & 0 \le x < t \\ 1 & t \le x \end{cases}$$

- Mean total life

$$E[\beta_t] = E[\gamma_t] + E[\delta_t]$$
$$= \frac{1}{\alpha} + \int_0^t P[\delta_t > x] dx$$
$$= \frac{1}{\alpha} + \int_0^t e^{-\alpha x} dx$$
$$= \frac{1}{\alpha} + \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right)$$

The mean total life is significantly larger than the mean life $E[X_k] = \frac{1}{\alpha}$ of any particular renewal interval. When *t* is large, the mean total life $E[\beta_t]$ is approximately twice the mean life.