Random Processes-Homework 2

1 Suppose that the social class of successive generations in a family follow a Markov chain with transition probability matrix given by

$$\text{Father's class}\quad \begin{array}{c} \\ L \\ M \\ U \end{array} \begin{array}{c} \text{Son's Class} \\ \begin{array}{ccc} L & M & U \end{array} \\ \left\|\begin{array}{ccc} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{array}\right\| \end{array}$$

(1) What fraction of families are upper class after 20 generations?

(2) What fraction of families are upper class in the long run?

2 A Markov chain has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\|\begin{array}{ccc} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0 & 1 \end{array}\right\| \end{array}$$

The Markov chain starts at time zero in state X_0=0. Let $T = \min\{n \geq 0: X_n=2\}$ be the first time that the process reaches state 2. Determine $P\{X_3 = 0 \mid X_0 = 0, T > 3\}$.

3 A coin is tossed repeatedly until two successive heads appear.

(1) Find the mean number of tosses required.

(2) Determine the mean time to reach state 2 starting from state 0 by invoking a first step analysis.

4 Prove Equations (2-47), (2-48) and (2-49) in the lecture notes (Chapter_2.pdf).

5 Prove Equation (2-66) in Chapter_2.pdf.

6 Consider the random walk Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\|\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 \\ 0 & 0.1 & 0 & 0.9 \\ 0 & 0 & 0 & 1 \end{array}\right\| \end{array}$$

Starting in state 1, determine the mean time until absorption. Do this first using the basic first step approach, and second using Equation (2-64) in Chapter_2.pdf.

7 Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\|\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.6 & 0.1 \\ 0 & 0 & 0 & 1 \end{array}\right\| \end{array}$$

(1) Determine the probability of absorption into state 0 starting from state 1.

(2) Determine the mean time spent in each of states 1 and 2 prior to absorption.

8 The possible states for a Markov chain are the integers 0, 1, …, N, and if the chain is in state j, at the next step it is equally likely to be in any of the states 0, 1, …, j–1. Formally

$$P_{ij} = \begin{cases} 1 & \text{if } i = j = 0 \\ 0 & \text{if } 0 < i \le j \le N \\ 1/i & \text{if } 0 \le j < i \le N \end{cases}$$

(1) Determine the fundamental matrix for the transient states 1, 2, …, N.

(2) Determine the probability distribution for the last positive integer that the chain visits

9 A Markov chain on the states 0, 1, 2, 3 has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 0.1 & 0.2 & 0.3 & 0.4 \\ 0 & 0.3 & 0.3 & 0.4 \\ 0 & 0 & 0.6 & 0.4 \\ 1 & 0 & 0 & 0 \end{array} \right\| \end{array}$$

Determine the corresponding limiting distribution.