Stochastic Hydrology 622 U3600

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Homework 2

1. The exponential density $f(x) = \lambda e^{-\lambda x}$ with λ=1 lies everywhere above the standard normal density $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ for $0 \le x < +\infty$. Show that the acceptance/rejection method with $g(x) = \sqrt{\frac{2e}{\pi}} e^{-x}$ can be used to generate random samples of the normal distribution.
2. Generate N = 10,000 random samples of Pearson type III distribution with coefficient of skewness $\gamma = 0.5,\ 1.0,\ 1.5$ and sample sizes n = 20, 40, 60, 80, 100, 150, 300, 500, 1000, 1500, 2000, respectively.

   (1) For each random sample, calculate its sample coefficient of skewness by

   $$\hat{\gamma} = \frac{n}{(n-1)(n-2)} \sum_{i=1}^{n} (x_i - \bar{x})^3 \Big/ s^3 \quad \text{where} \quad s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2}\text{, and}$$

   $$\hat{\gamma}' = \hat{\gamma} \frac{\sqrt{n(n-1)}}{n-2} \left(1 + \frac{8.5}{n}\right).$$

   (2) Construct a plot of $\hat{\gamma}$ and $\hat{\gamma}'$ vs. n and make your comment on the results.
3. Generate 100 random variates (i.e., a random sample of sample size n = 100) from each of the following distributions and calculate their sample means and sample variances.

   (1) Poisson distribution $f_X(x;\lambda) = \frac{e^{-\lambda} \lambda^x}{x!},\ x = 0,1,2,...$ with λ = 5.

   (2) Normal distribution $N(2.0, 4.0)$

   (3) X with density function $f_X(x) = 12x^2(1-x),\ 0 < x < 1.$
4. Let X be a random variable with the standard normal density. Estimates $P[X \le 0.78]$ by an appropriate simulation method.

   Integration using the rejection method for stochastic simulation
5. Devise a stochastic simulation method to estimate the numerical value of π. [Note: this is equivalent to estimate the integration of $f(x) = 4\sqrt{1-x^2}$ in the [0,1] interval.]
6. Any object with temperature above absolute zero is a source of electromagnetic

radiation. However, the spectral composition and magnitude of spectral radiant emittance vary with the absolute temperature of the object. Such phenomenon can be explained by the Planck's Law, which states that

$$M_\lambda(T) = \frac{2\pi hc^2\lambda^{-5}}{e^{ch/\lambda kT}-1} \tag{1}$$

where

M_λ = wavelength-dependent spectral radiant emittance (Joules/sec/m^2/m)
λ = wavelength (m)
h = Planck's constant, 6.6256×10^{-34} Joules-sec
c = the speed of light, 3×10^{8} m/sec
k = Planck's constant, 1.38×10^{-23} Joules/°K
T = absolute temperature (°K)

The concept of spectral radiant emittance M_λ is not the same as the radiation energy, and it is better to be perceived as the differential function of the total energy M emitted by an object, with respect to wavelength λ. The spectral radiant emittance to radiated energy in remote sensing is analogous to the probability density to cumulative probability in statistics.

The total emitted energy M is obtained by integrating M_λ over the full range of spectral wavelength, i.e.

$$M = \int_0^\infty M_\lambda(T)d\lambda = \int_0^\infty \frac{2\pi hc^2\lambda^{-5}}{e^{ch/\lambda kT}-1}d\lambda \tag{2}$$

Let $x = ch/\lambda kT$ and through change of variable, we have

$$M = \frac{2\pi(kT)^4}{h^3c^2}\int_0^\infty \frac{x^3}{e^x-1}dx \tag{3}$$

It can be shown that $\int_0^\infty \frac{x^3}{e^x-1}dx = \frac{\pi^4}{15}$, and thus the total energy emitted by a blackbody of absolute temperature T equals

$$M = \frac{2\pi(kT)^4}{h^3c^2}\int_0^\infty \frac{x^3}{e^x-1}dx = \sigma T^4 \tag{4}$$

where $\sigma = 5.6697\times10^{-8}$ W/m^2/°K^4.

The above equation is known as the Stefan-Boltzmann law and it states that the magnitude of total energy a blackbody can emit is proportional to the fourth power of its absolute temperature. However, we sometimes need to calculate the amount of energy emitted by a black body over a wavelength range $\Delta\lambda = (\lambda_1,\ \lambda_2)$, i.e.

$$M_{\Delta\lambda}(T) = \int_{\lambda_1}^{\lambda_2} M_\lambda(T)d\lambda = \int_{\lambda_1}^{\lambda_2} \frac{2\pi hc^2\lambda^{-5}}{e^{ch/\lambda kT}-1}d\lambda \tag{5}$$

The integral cannot be calculated analytically.

Tasks

(1) Estimate the daily energy emitted by a blackbody of surface temperature 300°K over the 6μm ~ 30μm wavelength range using the rejection method for stochastic simulation. Assume the blackbody is a sphere with a radius r = 1 km.

(2) Estimate the daily energy emitted by the same blackbody over the 480μm ~ 800μm wavelength range.

(3) If $\frac{ch}{\lambda kT} << 1$, it can be shown that $M_{\lambda}(T) \approx \frac{2\pi hc^2}{\lambda^5 (ch/\lambda kT)} = \frac{2\pi ckT}{\lambda^4}$. This is known as the Rayleigh-Jean's approximation. Compare your estimate in Task (2) against the estimate by the Rayleigh-Jean's approximation.